Making The Pendulum Isochronous

Huygens' analytical approach to making the pendulum isochronous was to consider the motion of a particle moving along a plane curve in space under the influence of gravity, as in Figure B1. We can look at the problem in the same way using our now standard mathematical techniques which we have inherited from the late C17th. We add, for completeness, a set of forces on the particle with a resultant having components T_p and T_n along and normal to the path respectively

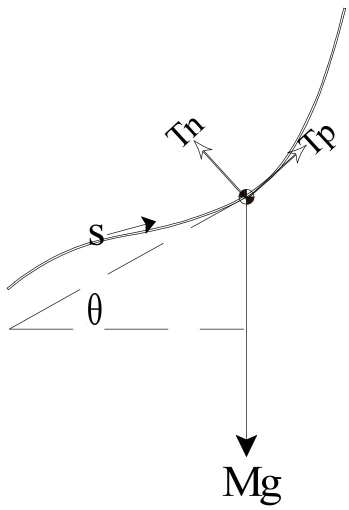

Fig B1

Referring to Figure B1, in which s is the distance along the curve and θ is the angle between the tangent to the curve and the horizontal. The motion will be a simple harmonic oscillation [86], and therefore isochronous if

$$\frac{d^2 s}{dt^2} \propto -s \qquad \text{B1}$$

Considering the change in displacement of the particle and resolving the acceleration due to gravity along the path, the equation of motion is

$$\frac{d^2 s}{dt^2} = -g\sin\theta + \frac{T_p}{M}$$

Comparing with equation B1, the motion will be simply harmonic if

$$s \propto g\sin\theta - \frac{T_p}{M}$$

that is if

$$s = C(g\sin\theta - \frac{T_p}{M})$$

Thus if the motion were simply harmonic we would have

$$\frac{ds}{d\theta} = C(g\cos\theta - \frac{1}{M}\frac{dT_p}{d\theta})$$

Note that while this is a sufficient condition for isochronism it is not a necessary condition. There may be some other path that is isochronous but not a simple harmonic motion.

If T_p does not vary with θ , and in particular if $T_p = 0$ as will be seen to be the case in Huygens arrangement

$$\frac{ds}{d\theta} = Cg\cos\theta \qquad \text{B2}$$

To examine what path the particle must follow for equation B2 to apply, establish a Cartesian co-ordinate system as shown in Figure B2. We wish to know how the x and y co-ordinates of the particle should vary as θ changes.

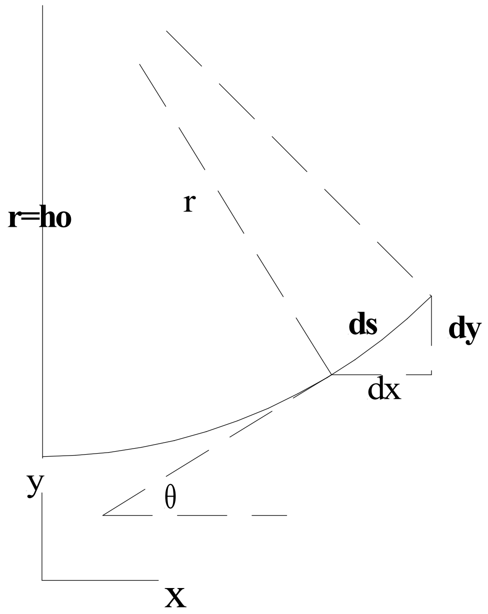

Figure B2

From an initial displacement s, infinitesimally displace the particle through ds, and denote the corresponding changes in x and y as dx and dy as again shown in Figure B2.

Then: $\frac{dy}{ds} = \sin\theta$, $\frac{dx}{ds} = \cos\theta$, and, for the motion to be simple harmonic, $\frac{ds}{d\theta} = Cg\cos\theta$

Thus $\frac{dx}{d\theta} = \frac{dx}{ds}\frac{ds}{d\theta} = Cg\cos^2\theta = \frac{Cg}{2}(1+\cos 2\theta)$ and

$$\frac{dy}{d\theta} = \frac{dy}{ds}\frac{ds}{d\theta} = Cg\sin\theta\cos\theta = \frac{Cg}{2}\sin 2\theta$$

Integrating with respect to θ , and setting constants of integration so that x = y = 0 at θ = 0 gives:

$$x = \frac{Cg}{4}(2\theta + \sin 2\theta) \qquad \text{and} \qquad y = \frac{Cg}{4}(1 - \cos 2\theta) \qquad \text{B3}$$

These are the equations of a cycloid with the cusps pointing upwards in which the rolling circle has radius $\frac{Cg}{4}$, and has turned through 2θ. The radius of the rolling circle is not arbitrary.

This derivation is applicable not only to a particle but also to the centre of mass of a rigid body of any shape.

Huygens' principle was then that, under the action of a constant vertical force, the motion of the centre of mass will be simply harmonic and therefore isochronous if it follows a particular cycloidal path:

because, on the particular cycloidal path $\frac{ds}{d\theta} = Cg\cos\theta$ so that $s \propto -g\sin\theta$ and therefore under a constant vertical force $\frac{d^2s}{dt^2} \propto -s$, which is a sufficient condition for isochronism

(where θ is the instantaneous inclination of the path to the horizontal axis of the cycloid.)

Huygens' application of cycloidal plates satisfied his principle because of two geometric facts:

The involute of a cycloid is itself a cycloid, and, accordingly, the end of a taut string unwrapping from a cycloidal chop will follow a cycloidal path.

The tangent to any evolute is normal to the involute at their intersection and thus the taut string of Huygens' pendulum was normal to the path of the centre of mass and the tension in it could have no component along the path. So $T_p = 0$. The acceleration along the path was then that due to the vertical gravity force alone.

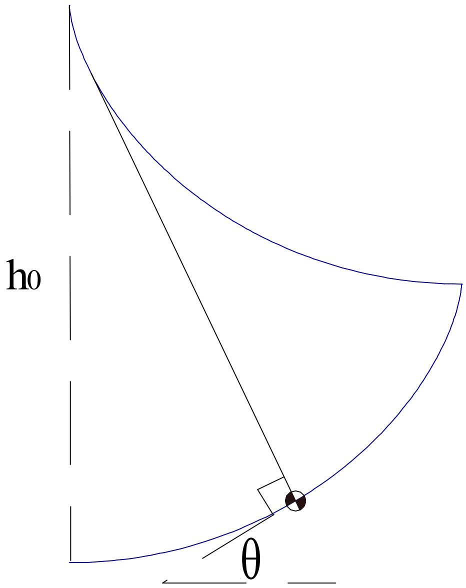

Figure B3 Huygens Arrangement

Moreover, Huygens' taut string lies instantaneously along the radius of curvature of the path of the bob. If the path is to be that followed by the bob of a simple pendulum initially suspended a distance h_o above the origin of the Cartesian co-ordinates then, when $\theta = 0$, the radius of curvature $r = h_0$

By definition, curvature = $\frac{d\theta}{ds}$ and consequently the radius of curvature $r = \frac{ds}{d\theta}$

For simple harmonic motion , from equation B2, $\frac{ds}{d\theta} = Cg\cos\theta$

so that for simple harmonic motion $r = Cg\cos\theta$ B4

Substituting $r = h_0$ when $\theta = 0$ in B4 leads to $Cg = h_0$ and from B3 the parametric equations of the cycloidal path for Huygens' pendulum are then

$x = \frac{h_0}{4}(2\theta + \sin 2\theta)$ and $y = \frac{h_0}{4}(1 - \cos 2\theta)$ B5 and we note that the radius of the generating circle is one quarter of the length of the pendulum string.

Standard texts show that the period of the cycloidal pendulum is given by $T = 4\pi\sqrt{\frac{a}{g}}$ where a is the radius of the generating circle. In Huygens' arrangement $a = \frac{h_0}{4}$ so that $T = 2\pi\sqrt{\frac{h_0}{g}}$ as for the simple pendulum at small angles.

This method of considering motion along the path using infinitesimal calculus was not available to Huygens who was working well before Newton devised and published his methods. Huygens had available the notions of mechanics that became Newton's first two laws of motion and the methods of analytical geometry. Generally speaking, geometrical methods do not permit the shape of a curve to be inferred from its other characteristics.

A bookseller's review of Traite de la Pendule a Cycloide says that in about 1684 the author of the book P. Baert independently reached the same conclusions as Huygens using different methods.

In discovering that the isochronous pendulum needed the cycloid Huygens was perhaps lucky. To emphasise this point, consider that the complementary problem, [87] that of finding the curve which was a brachistochrone, required infinitesimal calculus and was not solved until 1697. But, it was Huygens' perspicacity which earned his luck.

The Rigid Body Pendulum

Huygens had briefly investigated the rigid body pendulum in response to a request from Mersenne in 1646. He was not able to achieve a result and dropped the inquiry[88]. When Huygens began to regulate the clocks using a supplementary weight sliding on the pendulum rod, the question of the rigid pendulum became more relevant.

Huygens investigated the behaviour of the rigid pendulum, under the heading "centre of oscillation" an historical term of confusing meaning which persisted through the C20th. It was a term used by mathematicians of the time and useful from Huygens' perspective because it permits one to find a two-mass system that is dynamically equivalent to the rigid body. Without the assistance of the integral calculus, he identified, and showed how to calculate, what we now know as the moment of inertia and centre of gravity and thereby determined the period of what we have come to call the compound pendulum. The relevant work notes are dated August to November 1661. [89] These notes became Part IV of *Horologium Oscillatorium*.

Huygens' Pendulum

Consider now Huygens' real pendulum as generalised in Figure B4. A rigid body, centre of mass G is pin jointed at another point A to a string AO fixed at O and unwrapping around a curved plate OP. The string is instantaneously tangential to the plate at O'.

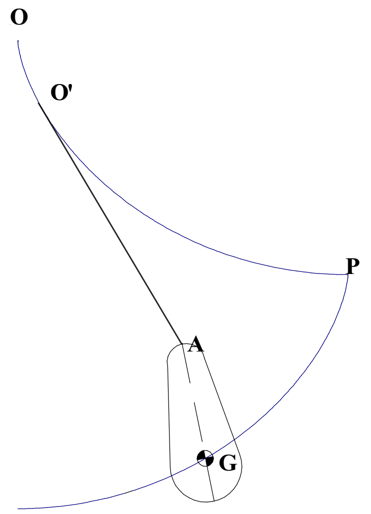

Figure B4

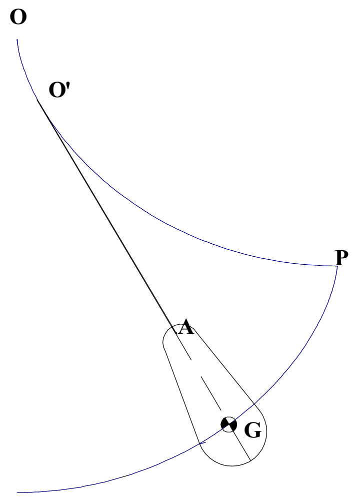

Figure B5

According to the definition of an involute, if G is to move along the involute to OP , then O' A and G must always be collinear as shown in Figure B5. This means that the rigid body must continuously rotate to remain aligned with the string as it wraps and unwraps.. The rate of rotation falls to zero and reverses at the end of each swing and is elsewhere generally not constant. To provide this acceleration the resultant of the forces acting on the body must have a moment about G.

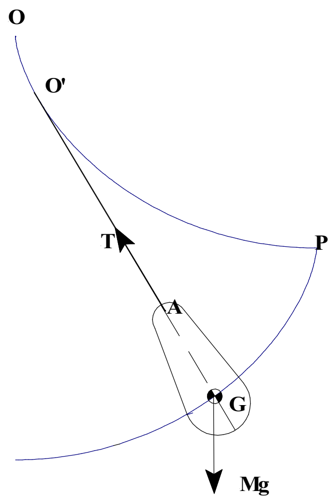

Figure B6

The forces acting on the body are Mg due to gravity and T the tension in the string as shown in Figure B6. If O' A and G are collinear , as shown, these forces have no moment about G, the angular acceleration of the body is zero and the colinearity of O' A G cannot be maintained.

Thus no simple arrangement of a string wrapping around a cheek will cause the centre of mass of the rigid body to follow the involute of the cheek. [1]

Huygens was well aware[90] of this by the end of 1661. It seems likely that Huygens' demurrer went unnoticed, for the unqualified acceptance of the cycloidal path as isochronous went on in horological circles for some considerable time.

If by some arrangement the centre of mass of a rigid body suspended from a string were persuaded to traverse a

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1 Unless A and G are coincident. That is, unless the bob is suspended freely from a pin joint at its centre of mass.

cycloid, we would have $\frac{ds}{d\theta} = k\cos\theta$ as a property of any cycloid and thence $s = -k\sin\theta + j$ (where k and j are constants).

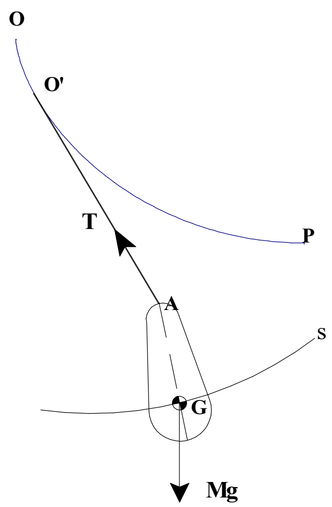

Figure B7

Since, as we have shown, O' A and G cannot remain collinear, the forces on the rigid body will be as shown in Fig B7 The acceleration along the path will be given by:

$$\frac{d^2s}{dt^2} = -g\sin\theta + f(T)$$

but $$s = -k\sin\theta + j$$

so that $$\frac{d^2s}{dt^2} \neq -s \times \text{ constant}$$

and the motion is therefore not simple harmonic.

Thus, if a rigid body is suspended by a string and its centre of mass traverses a cycloid, the motion of the centre of mass will not be isochronous.

There remains the possibility that, despite the centre of mass not following the involute of the cheek, there might be some shape of cheek which causes the motion to be isochronous.

Other clockmakers have been seen to use curved chops to alter the swing of the pendulum notably Arnold and Harrison, suggesting that the principle might work after all. Closer examination shows that in the long run Harrison used chops not to impart isochronism by their shape, but to provide an adjustment mechanism to minimise the variation in rate caused by extraneous influences.

In 1818 Benjamin Gompertz showed that the cycloid was not isochronous for a rigid pendulum and attempted to derive the correct isochronous path. He concluded that the objective could not be achieved by a pendulum's suspension cord wrapping around cheeks.[91]. His work appears to have gone unnoticed by horologists.

Proposition XXIV

Huygens demurrer about the cycloidal pendulum appears in *Horologium Oscillatorium* Part IV Proposition XXIV.[92] As translated he says." It is not possible to determine the centre of oscillation for pendula suspended between cycloids."

What was Huygens' authority for this proposition? Unlike most of the other Propositions there is no supporting geometric or kinematic argument. Perhaps the problem was that that the centre of oscillation is continuously changing as the string wraps around the chops. But Huygens refutes that. In Proposition XXIV he says, *"Posset enim videri, etiam centrum oscillatlonis. mutari, ad singulas diversas longitudines; quod tamen hoc modo intelligendum non est. Res sane explicatu difficillima ..."* "[For it would seem that its centre of oscillation changes for each different length. But this should not be understood in that way. The matter is most difficult to explain …]"

To determine the centre of oscillation we must first locate the centre of rotation. The centre of rotation will be moving so we adopt the concept of the instantaneous centre of rotation.

At any particular time, the complete motion of a rigid body can be represented as a rotation of the body as a whole about an instantaneously fixed point in space - the instantaneous centre of rotation. The velocity of any point in the body is proportional to the distance from the point to the instantaneous centre; and the direction of the velocity is perpendicular to the line joining the point to the instantaneous centre. We adopt the following notation:

$v_A =$ the velocity of point A , $v_{BA} =$ the relative velocity of point B with respect to point B

and we recognise that $v_B = v_A + v_{BA}$. Further, we note that if A and B are two points fixed in a rigid body a distance *r* apart $v_{BA} = r\omega$, where ω is the angular velocity of the body, and v_{BA} is perpendicular to the line joining A to B. Figure B8 illustrates and shows how the velocity of B may be found if the rotational velocity and the velocity of A are known. $v_B = v_A + r\omega$ With that information the instantaneous centre of rotation of the body may be found. It lies at the intersection of the two lines BP and AQ drawn perpendicular to v_B and v_A respectively as shown in Figure B9.

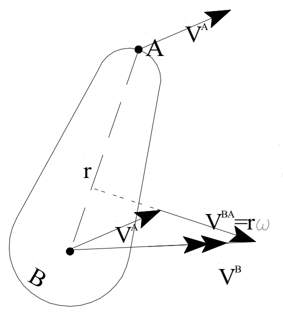

Figure B8

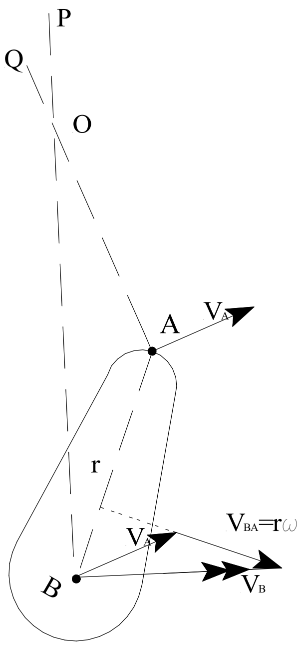

Figure B9

Consider now the rigid body of Figure B10 , with its centre of mass at B and suspended from a taut string of length L and fixed at Z. If the angular velocity of the string is Ω we now have $v_A = L\Omega$ and thence $v_B = L\Omega + r\omega$. The instantaneous centre of rotation again lies on AQ and on BP drawn perpendicular to v_B at B. The instantaneous centre of oscillation of the bob lies along OB extended and a distance $\frac{k^2}{h}$ beyond B. The position of the centre of oscillation changes continuously and depends upon the orientation and angular velocity of the string and the bob. It cannot be uniquely determined as it depends on the initial conditions.

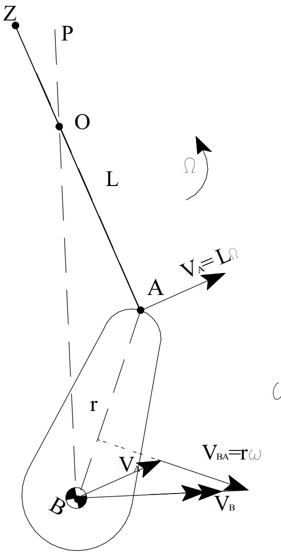

Figure B10

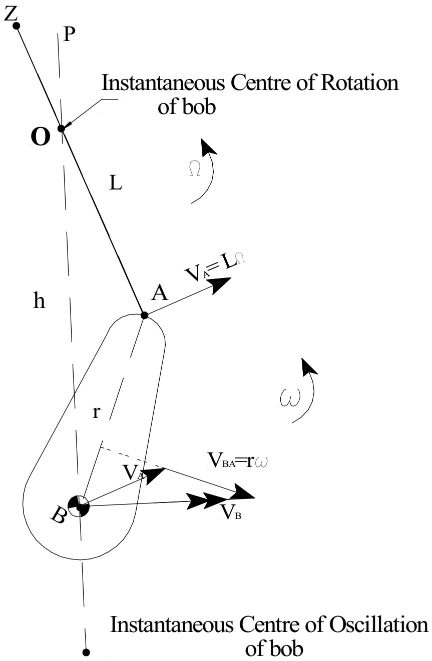

Figure B11

Proposition XXIV tells us that Huygens was aware that the string, bob, and rod did not swing as a rigid body. This meant that the bob would not follow the involute of the chops and the cycloids would not deliver isochronism. The significance of Prop XXIV is obscure until one makes this connection.

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86 Indeed this principle itself is due to Huygens. He enunciated the principle in 1673 calling it. *incitation parfaite decroisante* (perfectly reducing impulsion ?)

87 Given two points, such that the straight line joining them is neither vertical nor horizontal, find how a curve between them must be drawn if a particle falling along the curve under gravity is to reach the bottom of the curve in the least possible time.

88 Blackwell R.J op cit p107, and
Mahoney M.S translator, Christiaan Huygens, The Pendulum Clock Part 4 On the Centre of Oscillation, 1997, 1995 http://www.princeton.edu/~hos/mike/texts/huygens/centosc/huyosc.htm

89 *Oeuvres Complètes* Vol XVI pp 414 Note 2

90 *Horologium Oscillatorium* Part IV Proposition XXIV

91 Gompertz B. On Pendulums Vibrating Between Cheeks, in The Journal of Science and the Arts No V Vol III, pp13- 33, The Royal Institution of Great Britain , James Eastburn & Co, New York, 1818

For this reference I am grateful to Fortunat Mueller-Marrki of the NAWCC

92 *Horologium Oscillatorium OC* Vol XVII p347