**UNRESTRAINED SIMPLE PENDULUM ON ROTATING EARTH**

This paper is an attempt to set out the Cartesian equations of motion of an otherwise simple pendulum, that is not restrained to swing in one plane, suspended over a spherical rotating earth.

In particular it is an attempt to set down the fully generalised equations rather than the approximate and two dimensional equations normally set out in text books. The purpose is to provide a basis for examining changes in the instantaneous plane of swing from North South to East West orientation and corresponding changes in the oscillatory motion.

## System of Axes

A line drawn from the point of suspension to the centre of the Earth O intersects the surface of the Earth at P.

The origin of the co-ordinate system is O. The z axis lies along OP. The axis OY is parallel to the North direction at P. The x axis is therefore parallel to the East direction at the point P.

This axis system is fixed to the Earth and therefore rotates in inertial space with the angular velocity of the Earth $\Omega$.

The pendulum string is of length $\boldsymbol{l}$ and is suspended a distance $\boldsymbol{l}$ above the surface along OP extended.

The latitude of P is $\lambda$.

The position vector of the pendulum bob in this system of axes is **R** with components **x, y, z**

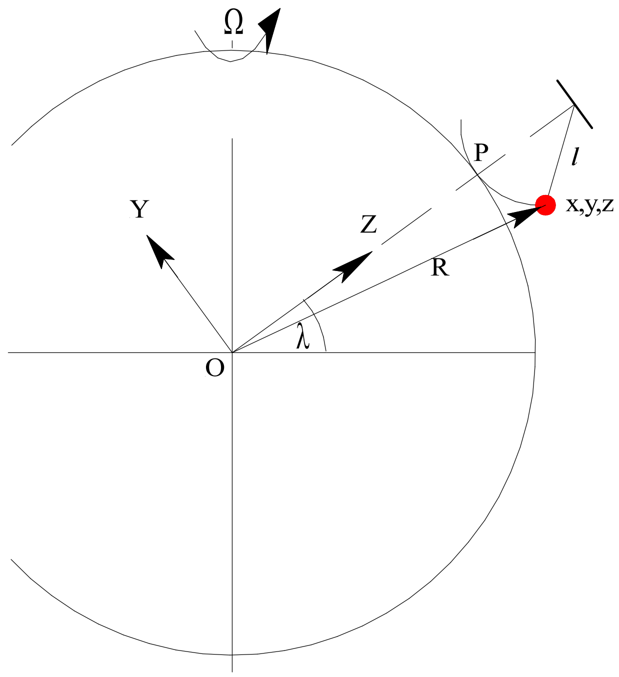


Figure 1 Axes System

**Absolute Acceleration**

The absolute acceleration of a particle moving with respect to a rotating frame of reference is:

$$\ddot{R} + \dot{\Omega} \times R + 2\Omega \times \dot{R} + \Omega \times (\Omega \times R)$$

In this case Ω is constant so that

$$\ddot{R} + 2\Omega \times \dot{R} + \Omega \times (\Omega \times R)$$

Transforming into Cartesian co-ordinates:

<u>Translation</u>

$\ddot{R} =$

$\ddot{x}$ in directionOX

$\ddot{y}$ in direction OY

$\ddot{z}$ in direction OZ

<u>Coriolis</u>

Ω has components $\varpi_x$, $\varpi_y$, $\varpi_z$ so that

$$2\Omega \times \dot{R} = 2\left[\left(\omega_x + \omega_y + \omega_z\right) \times \left(\dot{x} + \dot{y} + \dot{z}\right)\right] =$$

$2\left(\omega_y \dot{z} - \omega_z \dot{y}\right)$ in direction OX

$2\left(\omega_z \dot{x} - \omega_x \dot{z}\right)$ in direction OY

$2\left(\omega_x \dot{y} - \omega_y \dot{x}\right)$ in direction OZ

But, assuming the Earth's orbital rotations are negligible, Ω is the earth's spin and resolving Ω gives:
$\omega_x = 0, \omega_y = \Omega \cos\lambda,$ and $\omega_z = \Omega \sin\lambda$

Thus Coriolis acceleration is:

$2(\Omega \cos\lambda \dot{z} - \Omega \sin\lambda \dot{y})$ in direction OX

$2(\Omega \sin\lambda \dot{x})$ in direction OY

$2(-\Omega \cos\lambda \dot{x})$ in direction OZ

Centrifugal

$$\Omega \times (\Omega \times R) = (\Omega \bullet R)\Omega - (\Omega \bullet \Omega)R$$

$(\Omega \bullet R)\Omega =$

$(\omega_x x + \omega_y y + \omega_z z)\omega_x$ in direction OX

$(\omega_x x + \omega_y y + \omega_z z)\omega_y$ in direction OY

$(\omega_x x + \omega_y y + \omega_z z)\omega_z$ in direction OZ

Again $\omega_x = 0, \omega_y = \Omega\cos\lambda,$ and $\omega_z = \Omega\sin\lambda$ giving:

$0$ in direction OX

$(\Omega\cos\lambda y + \Omega\sin\lambda z)\Omega\cos\lambda$ in direction OY

$(\Omega\cos\lambda y + \Omega\sin\lambda z)\Omega\sin\lambda$ in direction OZ

$(\Omega \bullet \Omega)R =$

$-\Omega^2 x$ in direction OX

$-\Omega^2 y$ in direction OY

$-\Omega^2 z$ in direction OZ

Collecting terms

$\ddot{x} + 2(\Omega\cos\lambda\dot{z} - \Omega\sin\lambda\dot{y}) - \Omega^2 x$ along OX

$\ddot{y} + 2(\Omega\sin\lambda\dot{x}) + (\Omega\cos\lambda y + \Omega\sin\lambda z)\Omega\cos\lambda - \Omega^2 y$ along OY

$\ddot{z} + 2(-\Omega\cos\lambda\dot{x}) + (\Omega\cos\lambda y + \Omega\sin\lambda z)\Omega\sin\lambda - \Omega^2 z$ along OZ

Simplifying

$\ddot{x} + 2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) - \Omega^2 x$ along OX

$\ddot{y} + 2\Omega\dot{x}\sin\lambda + \Omega^2\left(y\cos^2\lambda + z\cos\lambda\sin\lambda - y\right)$ along OY

$$\ddot{z} - 2\Omega\dot{x}\cos\lambda + \Omega^2\left(y\cos\lambda\sin\lambda + z\sin^2\lambda - z\right) \text{ along OZ}$$

But, $\cos^2\lambda + \sin^2\lambda = 1$ and accelerations become:

$$\ddot{x} + 2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) - \Omega^2 x \qquad \text{along OX}$$

$$\ddot{y} + 2\Omega\dot{x}\sin\lambda + \Omega^2\left(z\cos\lambda\sin\lambda - y\sin^2\lambda\right) \text{ along OY}$$

$$\ddot{z} - 2\Omega\dot{x}\cos\lambda + \Omega^2\left(y\cos\lambda\sin\lambda - z\cos^2\lambda\right) \text{ along OZ}$$

**Applied Forces**

The forces applied to the bob are the tension in the string $T$ with components $T_x T_y T_z$ and the gravitational force W.

With the usual notation W is $G\dfrac{mM}{R^2}$ and acts in the direction **-R**
and has the direction cosines $\hat{x}, \hat{y},$ and $\hat{z}$

$$R^2 = x^2 + y^2 + z^2 \qquad \therefore W = G\frac{mM}{x^2 + y^2 + z^2}$$

**Equations of Motion**

Accordingly, the equations of motion are:

$$\ddot{x} + 2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) - \Omega^2 x = \frac{T_x}{m} - G\frac{M}{x^2 + y^2 + z^2}\hat{x}$$

$$\ddot{y} + 2\Omega\dot{x}\sin\lambda + \Omega^2\left(z\cos\lambda\sin\lambda - y\sin^2\lambda\right) = \frac{T_y}{m} - G\frac{M}{x^2 + y^2 + z^2}\hat{y}$$

$$\ddot{z} - 2\Omega\dot{x}\cos\lambda + \Omega^2\left(y\cos\lambda\sin\lambda - z\cos^2\lambda\right) = \frac{T_z}{m} - G\frac{M}{x^2 + y^2 + z^2}\hat{z}$$

**East West Oscillation vs North South**

For a pendulum instantaneously swinging NS ie in the yz plane

$x = \dot{x} = \hat{x} = 0$ And as the string cannot exert a force on the bob out of the plane of swing $T_x = 0$ Thence:

$$\ddot{x} + 2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) \;= 0$$

$$\ddot{y} + \Omega^2\left(z\cos\lambda\sin\lambda - y\sin^2\lambda\right) \;= \frac{T_y}{m} \;- G\frac{M}{y^2 + z^2}\,\hat{y}$$

$$\ddot{z} + \Omega^2\left(y\cos\lambda\sin\lambda - z\cos^2\lambda\right) \;= \frac{T_z}{m} \;- G\frac{M}{y^2 + z^2}\,\hat{z}$$

For a pendulum instantaneously swinging in the EW ie xz plane

$y = \dot{y} = \hat{y} = T_y = 0$ so that:

$$\ddot{x} + 2\Omega(\dot{z}\cos\lambda) - \Omega^2 x \;= \frac{T_x}{m} - G\frac{M}{x^2 + z^2}\,\hat{x}$$

$$\ddot{y} + 2\Omega\dot{x}\sin\lambda + \Omega^2(z\cos\lambda\sin\lambda) = 0$$

$$\ddot{z} - 2\Omega\dot{x}\cos\lambda - \Omega^2\left(z\cos^2\lambda\right) \;= \frac{T_z}{m} \;- G\frac{M}{x^2 + z^2}\,\hat{z}$$

Now, the direction cosines $\hat{x}$, $\hat{y}$, $\hat{z}$ and the components of *T* are functions of *x, y* and *z* and I think it is a fair bet that not even the rockets-and-gyroscopes people have solved these equations. However, two observations can be made.

Firstly, the bob has an out of plane acceleration in both cases. These will cause the plane of oscillation to rotate with respect to the Earth's surface. - as we see in the Foucalt pendulum. These accelerations are given by:

$\ddot{x} + 2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) \;= 0$ and

$\ddot{y} + 2\Omega\dot{x}\sin\lambda + \Omega^2(z\cos\lambda\sin\lambda) = 0$ That is:

$\ddot{x} = -2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda)$ in the NS case, and

$\ddot{y} = -2\Omega\dot{x}\sin\lambda - \Omega^2(z\cos\lambda\sin\lambda)$ in the EW case.

These expressions are sufficiently different to suggest that the rate of rotation of the Foucalt pendulum is not uniform.

Secondly, the two sets of equations do not have the degree of symmetry that would suggest that the period of swing in the NS case would be the same as in the EW case. The most relevant equations are:

$\ddot{y} + \Omega^2\left(z\cos\lambda\sin\lambda - y\sin^2\lambda\right) \;= \frac{T_y}{m} \;- G\frac{M}{y^2 + z^2}\,\hat{y}$ and

$$\ddot{x} + 2\Omega(\dot{z}\cos\lambda) - \Omega^2 x = \frac{T_x}{m} - G\frac{M}{x^2 + z^2}\,\hat{x}$$

**Simplifying Approximation**

The assumption we make with most analyses of pendulums is that the gravitational force acts along the line PO , the z axis. (Making $\hat{x} = \hat{y} = 0$ and $\hat{z} = 1$)

We also assume that the gravitational force does not change because of the increase in length of R as the pendulum swings. So that

$W = G\frac{mM}{r^2}$ where r is the radius of the Earth.

The equations of motion then become:

$$\ddot{x} + 2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) - \Omega^2 x = \frac{T_x}{m}$$

$$\ddot{y} + 2\Omega\dot{x}\sin\lambda + \Omega^2\left(z\cos\lambda\sin\lambda - y\sin^2\lambda\right) = \frac{T_y}{m}$$

$$\ddot{z} - 2\Omega\dot{x}\cos\lambda + \Omega^2\left(y\cos\lambda\sin\lambda - z\cos^2\lambda\right) = \frac{T_z}{m} - G\frac{M}{r^2}$$

These equations actually apply to any particle subject to gravity and a force *T*.

**Tension in String**

Having made that approximation, it is practicable to further evaluate *T* in the case of a simple pendulum.

We introduce $\theta$ the angular displacement of the pendulum string from the *z* axis and $\alpha$ the angular amplitude of the pendulum's swing . ie $\dot{\theta} = 0$ when $\theta = \alpha$

The string provides the centripetal acceleration of the bob which causes it to move in a circular path relative to the xyz axes system. The string also reacts the weight of the pendulum

Thus $$T - mg\cos\theta = ml\dot{\theta}^2 \quad ①$$

Where $g = G\frac{M}{r^2}$

The angular momentum principle, with $I_o$ as the mass moment of inertia about O, gives:

$$I_o\ddot{\theta} = -mgl\sin\theta \quad ③$$

Considering conservation of mechanical energy;

$$\frac{1}{2}I_o\dot{\theta}^2 = mgl(\cos\theta - \cos\alpha) \quad ④$$

recognising that $\frac{m}{I_o} = \frac{1}{l^2}$ because $l$ is also the radius of gyration of the pendulum about O,

From ① and ④

$$\frac{T}{mg} = \cos\theta(1 + 2\frac{l^2}{l^2}) - 2\frac{l^2}{l^2}\cos\alpha$$

ie $$\frac{T}{m} = (3\cos\theta - 2\cos\alpha)G\frac{M}{r^2} \quad ⑤$$

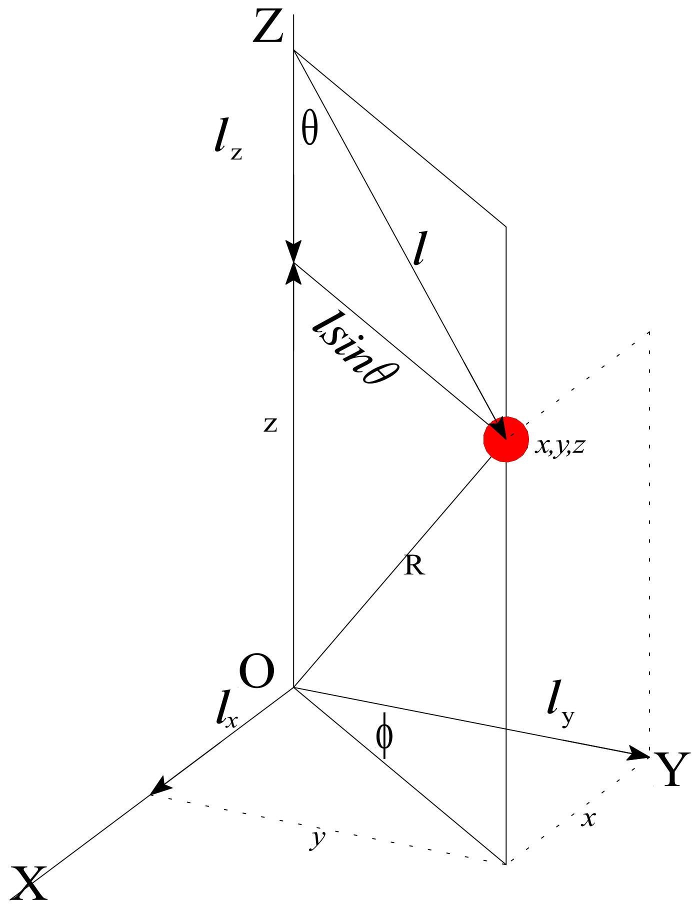


Figure 2 Components of $l$

$T$ and $l$ are colinear and in opposite directions so that from Fig2, noting also that $z+|l_z|=r+l$

$$T_{x=}=-T\sin\theta\sin\phi$$
$$T_y=-T\sin\theta\cos\phi$$
$$T_z=T\cos\theta$$

Where

$$\cos\theta=\frac{r+l\text{-}z}{l}$$
$$\sin\theta=\frac{\sqrt{x^2+y^2}}{l}$$
$$\cos\phi=\frac{y}{\sqrt{x^2+y^2}}$$
$$\sin\phi=\frac{x}{\sqrt{x^2+y^2}}$$

Thus

$$T_x=-T\frac{x}{l}$$
$$T_y=-T\frac{y}{l}$$
$$T_z=T\frac{r+l-z}{l}$$

But equation ⑤ gives

$$\frac{T}{m}=(3\cos\theta-2\cos\alpha)G\frac{M}{r^2}$$
$$=\left(3\frac{r+l-z}{l}-2\cos\alpha\right)G\frac{M}{r^2}$$

So that

$$\frac{T_x}{m}=-\left(3\frac{r+l-z}{l}-2\cos\alpha\right)G\frac{M}{r^2}\frac{x}{l}$$
$$\frac{T_y}{m}=-\left(3\frac{r+l-z}{l}-2\cos\alpha\right)G\frac{M}{r^2}\frac{y}{l}$$
$$\frac{T_z}{m}=\left(3\frac{r+l-z}{l}-2\cos\alpha\right)G\frac{M}{r^2}\frac{r+l-z}{l}$$

## Equations of Motion

The equations of motion now become

$$\ddot{x} + 2\Omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) - \Omega^2 x = -\left(3\frac{r+l-z}{l} - 2\cos\alpha\right)G\frac{M}{r^2}\frac{x}{l}$$

$$\ddot{y} + 2\Omega\dot{x}\sin\lambda + \Omega^2\left(z\cos\lambda\sin\lambda - y\sin^2\lambda\right) = -\left(3\frac{r+l-z}{l} - 2\cos\alpha\right)G\frac{M}{r^2}\frac{y}{l}$$

$$\ddot{z} - 2\Omega\dot{x}\cos\lambda + \Omega^2\left(y\cos\lambda\sin\lambda - z\cos^2\lambda\right) = \left(3\frac{r+l-z}{l} - 2\cos\alpha\right)G\frac{M}{r^2}\frac{r-z}{l}$$

## Further Approximations

Now $\Omega$ is small ( $= \dfrac{2\Pi}{86400} = 0.000073 \text{ radians/sec}$ ) and $z \approx r + l$ so that $\Omega^2 z \approx \Omega^2 r$

That implies that $\Omega^2 l$ is negligible. There are other terms of the same order that might also be called negligible, $\Omega\dot{z}$ is an example.

$\theta$ may also be said to be small so that $\cos\theta = 1$ and $\sin\theta = \theta$ and $r = z$

Applying successive approximations such as these will lead eventually to a set of equations of motion in which is solvable, but in which any differences between EW and NS oscillation disappear. In the context of the original question that seems a little pointless.

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